Transient State Analysis of a damped & forced oscillator

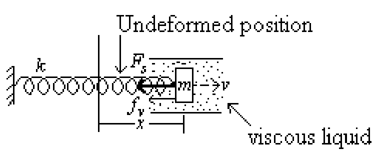
P.K.Sharma 1*

1. M-56, Flat No-302, Madhusudan Nagar, Jagannath complex, Unit-4, Odisha, India.

Abstract: This paper deals with the behaviour of an oscillator in its initial stage of oscillation. How are energy, displacement etc of the oscillator change with time? How does the phase difference between the driving force and velocity change with time in forced oscillation? Can we conserve energy of free damped and forced oscillation during first few seconds? How does the oscillator absorb energy from the supply when we drive it? How does the vibrating oscillator attain its steady state? How does an oscillation in steady state decay when the driving force is turned off? These all questions are answered in this paper.

Forced Oscillation with $F_{ext} = 0$

By a "transient" is meant a solution of the differential equation when there is no force present[1]. When we withdraw the oscillating driving force from the forced oscillator and do not neglect the friction, it drains the stored energy and damps the oscillation. For any deformation x of the spring of stiffness k, the net force acting on the block is, where and are the spring force and viscous (friction) force respectively.



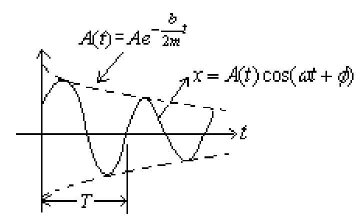
At the given position both spring force and friction opposes the motion

Or, $-kx - bv = m\dfrac{d^2x}{dt^2}$

Or, $\dfrac{d^2x}{dt^2} + \dfrac{b}{m}\dfrac{dx}{dt} + \dfrac{k}{m}x = 0$

If the damping constant b is smaller than the spring force, the solution of the above differential equation is given as $x = Ae^{-\frac{b}{2m}t}\cos(\omega t + x)$, where $A =$ maximum amplitude of oscillation and $\phi =$ arbitrary phase constant.

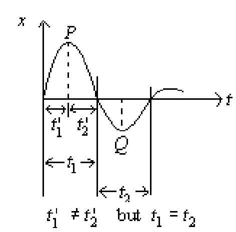
The coefficient of the harmonic term $\cos(\omega t + \phi)$ is given as $A(t) = Ae^{-\frac{b}{2m}t}$. Then, the above equation can be written as $x = A(t)\cos(\omega t + \phi)$,



The block m oscillates with decreasing amplitude for small damping

where $A(t)$ = amplitude of the oscillating block which decreases exponentially with time shown as the envelope of the displacement-time graph. We will see that the frequency of oscillation ω will be less than the undamped frequency ω_0.

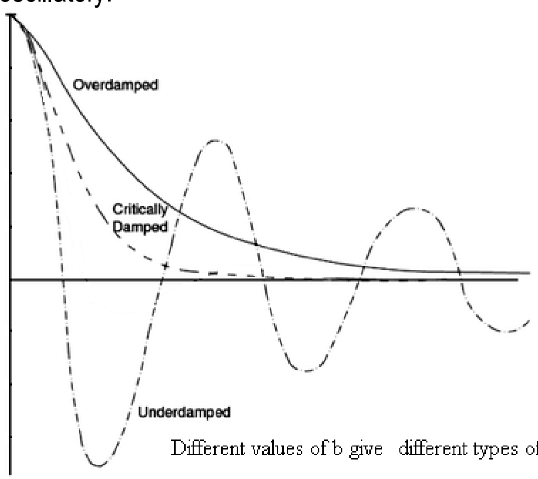
"Incidentally although the concept of a definite frequency can be strictly applied only to a pure sine or cosine function, ω is commonly called the frequency of oscillation. The zero crossings of the function.



$A(t) = Ae^{-\frac{b}{2m}t}\cos(\omega t + \phi)$ are separated by equal time intervals $T = \dfrac{2\pi}{\omega}$, but the peaks (P and Q) do not lie half way between them[2]." This means that $t_1 = t_2$ but $t_1' \neq t_2'(t_2' > t_1')$ as shown in the graph.

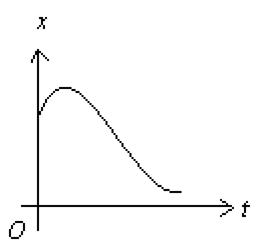
The block will oscillate with a frequency lesser than the free undamped frequency ω_0 due to the friction given as $\omega = \sqrt{\omega_0^2 - \frac{b^2}{4m^2}}$. Since, $\omega > 0$ for any value of $b < 2m\omega_0 \left(\omega_0 = \sqrt{\frac{k}{m}} \right)$, the block will oscillate with decreasing amplitude so many times before coming to rest. This is called under damping. If $b = 2m\omega_0$, the equation of motion of the block is given as $x = (At + B)e^{-\frac{b}{2m}t}$

This equation tells us that, the motion of the block is not oscillatory.



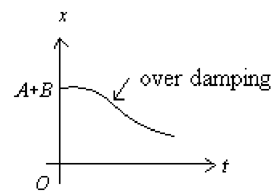
Different values of b give different types of motion

There is an initial rise in the displacement due to the factor $(At + B)$, where A and B are positive constants but subsequently the exponential term dominates as it increases further. The displacement can become zero for one finite value of time t. This situation is called critical damping[3].



For critical damping first 'x' increases then decreases to zero

For $b > 2m\omega_0$, due to heavy (over) damping the motion of the particle (block) of the spring-mass system is given by $x = Ae^{-b_1 t} + Be^{-b_2 t}$, where A and B are positive constants and $b_1 = \frac{b}{2m} - \sqrt{\frac{b^2}{4m^2} - \omega_0^2}$ and $b_2 = \frac{b}{2m} + \sqrt{\frac{b^2}{4m^2} - \omega_0^2}$ At $t = 0$, $x = A + B$ and if $t \to \infty$, $x \to 0$ as shown in the graph.



For large value of b (over damping) the block comes to rest without any oscillation

This physically signifies that when displaced, the block will come back to rest without executing any oscillation. This condition of over damping is also called "dead beat" or "a periodic displacement[4]".

Energy of a damped oscillator:

The total energy of the damped oscillator is $E = K + U$

$$= \frac{1}{2}mv^2 + \frac{1}{2}kx^2$$

$$= \frac{1}{2}m\left[\frac{d}{dt}\left\{Ae^{-\frac{b}{2m}t}\cos(\omega t + x)\right\}\right]^2 + \frac{1}{2}k\left[\frac{d}{dt}\left\{Ae^{-\frac{b}{2m}t}\cos(\omega t + x)\right\}\right]^2$$

$$= \frac{1}{2}m\left[-\omega Ae^{-\frac{b}{2m}t}\sin(\omega t + x) + A\left(-\frac{b}{2m}\right)e^{-\frac{b}{2m}t}\cos(\omega t + \phi)\right]^2 + \frac{1}{2}kA^2e^{-\frac{b}{m}t}\cos^2(\omega t + \phi)$$

For weak damping $b \cong 0$. Then the of the above expression will become zero. Hence, we can write

$$E = \frac{mA^2\omega^2}{2}e^{-\frac{b}{m}t}\sin^2(\omega t + \phi) + \frac{1}{2}kA^2e^{-\frac{b}{m}t}\cos^2(\omega t + \phi)$$

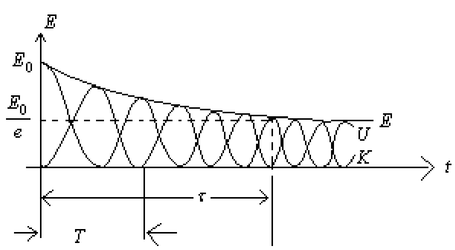
Since b is very small $\omega = \sqrt{\omega_0^2 - \frac{b^2}{4m^2}} \cong \omega_0$. Then,

$$E=\frac{mA^2\omega_0^2}{2}e^{-\frac{b}{m}t}\sin^2(\omega t+\phi)+\frac{1}{2}kA^2e^{-\frac{b}{m}t}\cos^2(\omega t+\phi)$$

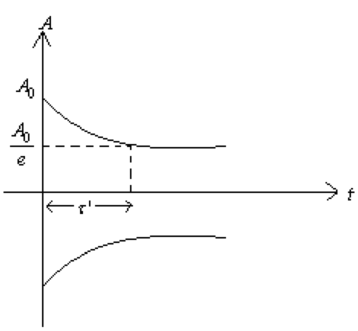
Putting $m\omega_0^2=k$ and taking $\frac{kA^2}{2}e^{-\frac{b}{m}t}$ common in both terms we have, $E=\frac{1}{2}kA^2e^{-\frac{b}{m}t}$, where $\frac{1}{2}kA^2=E_0=$ total energy of the oscillator at $t=0$. Then, we can write the variation of energy of the damped oscillator with time as, $E=E_0e^{-\frac{b}{m}t}$

Decay time & relaxation time:

This tells us that the total mechanical energy decays exponentially with a time constant $\tau=\frac{m}{b}$ due to the friction present in the oscillator. Time constant τ (or damping time or characteristic time of the oscillator) is defined as the time during which the energy of the damped oscillator decreases by "e" times, that is $E=\frac{E_0}{e}$.



$\tau >> T$ for weak damping, during time τ, energy decreases by "e" times



during time τ' amplitude decreases by "e" times

This must not be misinterpreted with the time 'τ'' during which displacement amplitude decreases by "e" times, because $\tau'=\frac{2m}{b}$, which is called "relaxation time" or "modulus of decay[5]". Both $\tau\,\&\,\tau'$ will be more if damping is less and vice versa.

Q-factor:

Quality factor (or Q-factor) of a damped oscillator is a measure of the rate at which the oscillator losses energy. The energy lost by the oscillator per second, that is, the power dissipated by friction is

$$\frac{dE}{dt}=P=-E_0\frac{b}{m}e^{-\frac{b}{m}t}$$

Then, the energy loss during one time period of oscillation, that is, one cycle is $\Delta E=\left|\frac{dE}{dt}\right|T$

$$=\frac{b}{m}E_0e^{-\frac{b}{m}t}.T$$

$$\Delta E=\frac{b}{m}ET$$

or, $\frac{E}{\Delta E}=\frac{b}{mT}$

Multiplying both sides by 2π, we have $2\pi\frac{E}{\Delta E}=\frac{2\pi}{T}\frac{m}{b}$

The LHS term is defined as the Q-factor of the oscillator as 2π times the ratio of energy of the oscillator at any instant and the energy lost (measured from that instant) during one cycle of oscillation[6]. Putting $\frac{2\pi}{T}=\omega$ and $\frac{m}{b}=\tau$, we have, $Q=\omega\tau=\frac{\omega}{2}\tau'\left(\because \tau'=\frac{\tau}{2}\right)$

For lightly damped oscillator, *b* is very less, hence τ is very large. So, Q is very large which physically signifies that the oscillator executes large number of oscillations to decay its energy (and amplitude) by a factor "e". For weak damping, $\omega\cong\omega_0\left(=\sqrt{\frac{k}{m}}\right)$ and $Q\cong\frac{k}{\sqrt{mb}}$. We can also mention the rate of energy loss, $\frac{dE}{dt} = -\frac{\omega E}{Q}$. If Q is more, energy decays less rapidly and vice versa.

Forced Oscillation with $F \neq 0$:

After we discussed the transients in a damped oscillation, let us apply this idea of transients in the initial stage of a forced oscillator before it attains a steady state. When a harmonic force $F = F_0 Cos\omega t$ acts on a damped oscillator of natural frequency $\omega_o = \sqrt{\frac{k}{m}}$, the net force acting on it is

$$F_{net} = F - kx - bv$$

Or, $m\frac{d^2x}{dt^2} + \frac{b}{m}\frac{dx}{dt} + \frac{k}{m}x = \frac{F_0}{m}\cos\omega t$

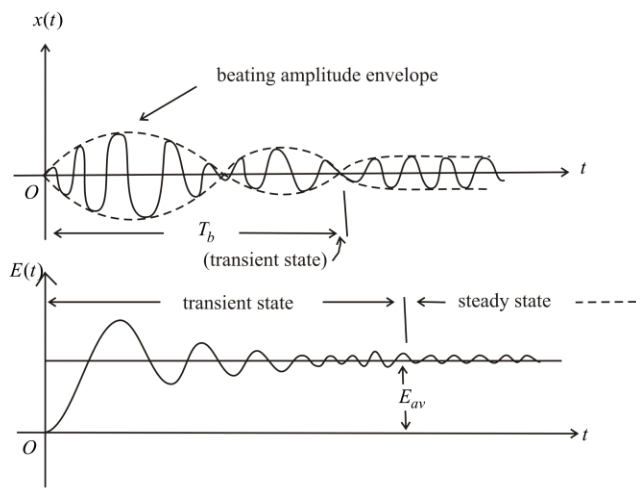
This differential equation has general solution

$x = x_p + x_c$, where x_p = particular integral = $ACos(\omega t + \theta)$ and x_c =complementary solution of the homogeneous equation $\frac{d^2x}{dt^2} + \frac{b}{m} + \frac{dx}{dt} + \frac{k}{m}x = 0,$ given as $x_c = Be^{\frac{-b}{2m}t}Cos(\omega t + \beta)$; $\omega = \sqrt{\omega_0{}^2 - \left(\frac{b}{2m}\right)^2} \cong \omega_0$ for weak damping, as discussed in earlier section.

Then, $x = ACos\left(\omega t + \phi\right) + Be^{-\frac{bt}{2m}}Cos\left(\omega t + \beta\right)$, where A, B, ϕ and β are constants to be calculated by using initial conditions[7]. After a long time $\left(t \to \infty\right)$ the second term, that is, transients will vanish and the steady state equation is established, given as

$$x = ACos(\omega t + \phi)$$

Initially the transients (short lived oscillations) given by the second expression of the general solution will be present after a long time (a time much greater than the mean time or critical time τ of the damped oscillation), all transient oscillations (beats) will vanish. Eventually the oscillation amplitude will be steady after a long time $t >> \tau$. As, at $t = 0,$ the block was at rest at the equilibrium position and in the due course of time the oscillator absorbs energy from the driving agent, its displacement amplitude will increase from zero to a maximum value; before it attains a steady state, the amplitude will oscillate several times with a frequency which is equal to half of the beat frequency $f_b = \frac{\omega - \omega_0}{2\pi}$ Since damping and a driving forces are there, the amplitude of oscillation will decrease to a constant value instead of going to zero.



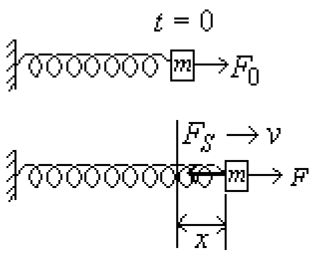
$x - t$ and $E - t$ graph with low damping near resonant frequency

[8] For weak damping and the driving frequency ω near $\omega_0\left(= \sqrt{\frac{k}{m}}\right)$, the energy of the oscillator can be given as the function of time as

$$E\left(t\right) \cong E\left[1 + e^{-\frac{b}{m}t} - 2e^{-\frac{b}{2m}t}\cos 2\left(\omega - \omega_0\right)t\right]$$

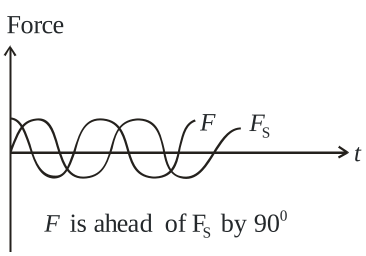
The energy graph tells us that during few mean times from starting (t=0) , the energy of the oscillator increases from zero, reaches the maximum and comes down to its minimum value alternatingly maintaining a time period called beat time $T_b = \frac{2\pi}{\omega - \omega_0}$. Further more , we can see from the graph that in each oscillation the maximum energy decreases with time to a uniform average value after a long time (practically few beat periods because T_b is large as $\omega \square \omega_0$)

Let us analyze the situation as following. At time $t = 0,$ the block was at rest. Hence, initially the kinetic energy is zero. Since the spring was undeformed initially, the initial potential energy stored in the spring is zero. This means that the total mechanical energy of the oscillator is zero initially. For the sake of simplicity let us ignore any friction. As the applied force continues to act toward right (say) for a time interval (from $t = 0$) $\Delta t = \frac{2\pi}{4\omega}$, it stretches the spring by a distance x, say.



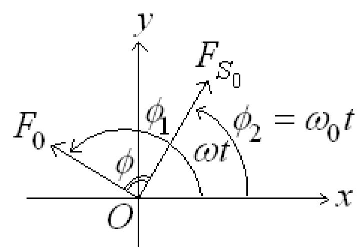
The spring force F_S opposes the driving force F at the given postion

Then the spring pulls the block back in the a spring force which is linear with x, gives as $F_s = kx$ Since x changes simple harmonically, the spring force will also change simple harmonically like the applied (driving) force F.



This means that two simple harmonic forces act on the block simultaneously but they do not become maximum (or minimum) at a time. Initially the driving force F is maximum (that is F_0) and the spring force F_s was zero. Thus F heads F_s by 90^0 at $t = 0$. This is the initial phase difference between these two forces; $\phi_0 = \frac{\pi}{2}$.

After a time t, the phase difference between these two forces is, $\phi = \phi_1 - \phi_2$, where ϕ_1 and ϕ_2 are the phases of the driving force and the spring force respectively after a time t. Since, $\phi_1 = \pi/2 + \omega t$ and $\phi_2 = \omega_0 t,$ we have $\phi = \pi/2 + (\omega - \omega_0)t$



The relative phase ϕ between F_0 & F_{S_0} changes with time

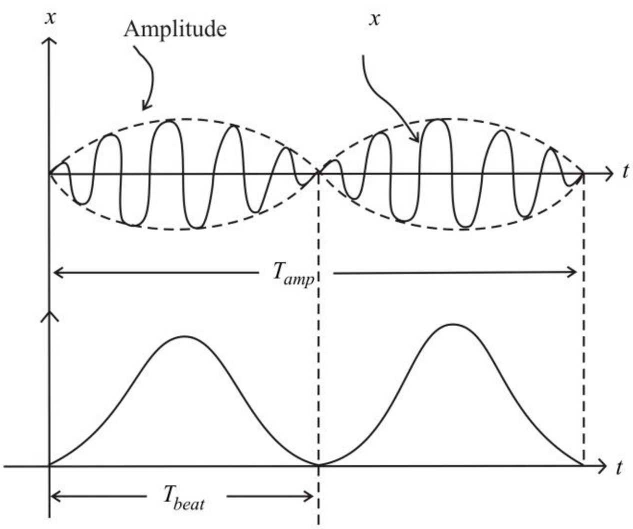
Initially, for sometime, the speed of the block will be small as it gains little kinetic energy from the external agent. Moreover, we assumed a small damping constant. Hence the effect of friction can be ignored for a small time internal (few time constants). Then the external force F will be added vertorially with the spring force to yield the net force.

$$F = \sqrt{F_{sp_0} + F_{ext_0} + 2F_{sp_0}F_{ext_0}\cos\phi},$$

where $\phi = \frac{\pi}{2} + (\omega - \omega_0)t, \quad F_{sp_0} = kA$ and $F_{ext_0} = F_0$

This is quite evident that the net force will fluctuate between maximum $F_{sp_0} + F_{ext_0}$ and minimum $\left|F_{sp_0} - F_{ext}\right|$ depending on the relative phase difference ϕ. The time period of fluctuation (alteration) of the net force acting on the block is $\Delta t = \frac{2\pi}{2|\omega - \omega_0|}$ (because at times $t = \frac{\left(2n - \frac{1}{2}\right)\pi}{|\omega - \omega_0|}$, where n =1,2,3…………, the forces are in phase or net force is maximum). Then the beating time (the time period of fluctuation of energy) is twice the period of alteration of force because energy is directly proportional to the square of the force, which is given as $T_b = \dfrac{2\pi}{2(\omega - \omega_0)}$.

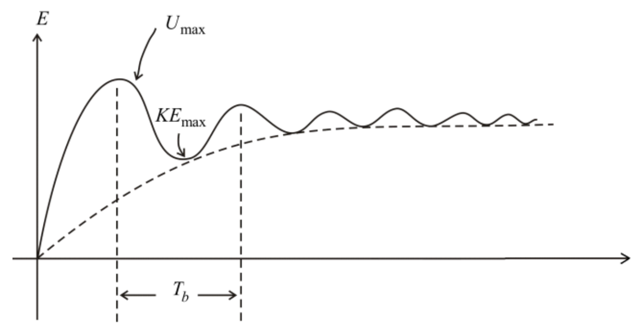
We have over simplified the argument by neglecting the friction for some time from starting. However, as the special increases, friction $f(=-bv)$ becomes more which will adjust the phase ϕ as we will discuss it later on. The motion will be more when the forces act in phase which increases the speed (KE) of the block in the first half of the beat period .This means that " The applied or driving force some times (for first half of T_b) is pushing with a relative phase "ϕ" which helps build up the oscillation amplitude, but it sometimes (second half of the beat time T_b) pushing with the opposite phase, thus diminishing the oscillations[9]".



Amplitude oscillates once in $\Delta t = T_{amp}$ and energy oscillates once in $\Delta t' = T_{beat} = \frac{T_{amp}}{2}$

As the time goes on, the work done by the driving force will be more, and hence the kinetic and potential energy of the oscillator increases from zero (through several maximum and minimum). When the displacement of the block will be maximum, the kinetic energy will be zero. At that time the total mechanical energy of the oscillating system is $U_{max} = \frac{1}{2}kA^2$. Similarly when the speed of the block is maximum, deformation of the spring will be zero and the kinetic energy becomes maximum given as $K_{max} = \frac{1}{2}mv_{max}{}^2$. Here, we cannot equate U_{max} with K_{max} due to two reasons; first of all the oscillating system is not purely conservative as friction is present, and secondly the driving force is being acted on the system from out side (external force). However, we can write the work energy theorem, as

$$W_{friction} + W_F = \Delta E_{system} = \Delta U + \Delta K$$



Total energy of the oscillator varies from zero to an average value if $\omega \neq \omega_0$

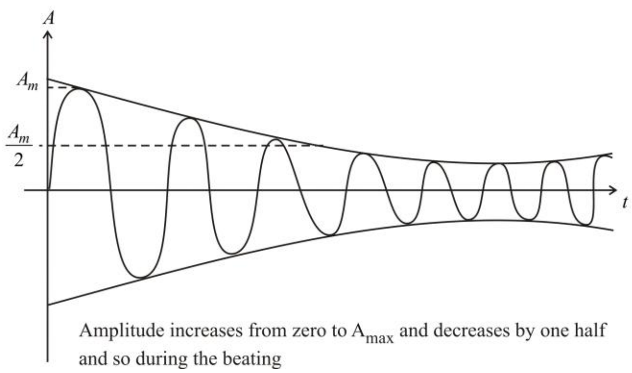
As the friction increases with the increasing speed, work done by friction increases. When the speed is maximum, maximum frictional work is done. Hence the maximum energy decreases since $W_{driving} = Fv$ and $W_{friction} = Fv = -bv^2 (\because f = -bv)$, for small value of v, $Fv >> bv^2$. Hence the energy increases from zero to some steady value (via ups and down of the total mechanical energy of the oscillator). At steady state, the energy E fluctuates between its maximum and minimum value through an average value.

However, for some time from starting, during which the total mechanical energy increase, and hence $\frac{dE}{dt} > 0$ as the driving force is performing a positive work. During each beat period the "total mechanical energy peak" decreases exponentially as displacement amplitude decreases exponentially due to frictional effect. As a result the energy beats will decay. In other words, the initial oscillation due to the spring force will damp due to friction. In this sense we can say that, because of damping the oscillator gradually adjusts its phase with respect to the driving force. After a sufficiently long time the oscillator settles into a steady state of vibration with no beats, oscillating exactly at driving frequency ω". This tells us that during first few seconds

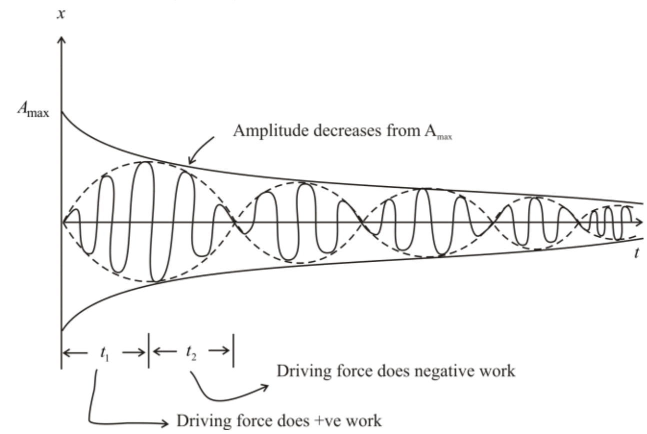
(τ_s) the oscillator absorbs energy which fluctuates between decreasing peaks and valleys. However, the average mechanical energy increases from zero to an average value $E_{av} = \frac{1}{4} m \ (\omega^2 + \omega_0{}^2) A^2$ after a long time, which is known as steady state. In steady state the relative phase difference between the driving force and velocity (or displacement or spring force) does not vary with time. Hence, we can say that in steady state.

"the amount of energy delivered to the oscillator in each push(each cycle) of the driving force is equal to the energy lost by the oscillator in one cycle due to frictional drag .Then the oscillator energy remains constant and the relative phase of the oscillator and driving force remains constant[11]". When the frequency of the driving force is equal to the natural or free undamped frequency of oscillation, that is $\omega = \omega_0$, the energy will build up (without any beats) smoothly to a steady value given by the expression. $E = E_0 \left(1 - e^{-\frac{t}{\tau}} \right)$.

Recapitulating, " Beats are damped due to the dominating effect of friction at higher speed ". We can see that the amplitude damps to about one half of the maximum value A_m it reaches at first when the beats are present[12]".



The driving force F may do some positive work when the block moves in the direction of F in the first t_1 seconds $\left(t_1 = \frac{\tau_b}{2} \right)$ and then does less negative work during time $\left(t_2 = \frac{\tau_b}{2} \right)$. As a whole it does a positive work increasing the total mechanical energy of the oscillating system inspite of the less negative frictional work. However, in steady state the work done by friction becomes significant as the speed increases whose average value, that is, average power dissipated by friction will be numerically equal to the average rate at which the driving agent is feeding the energy to the oscillatior. Mathematically, in steady state the average input power will be equal to the average power loss due to the friction and the average mechanical energy that the oscillator accumulates since the beginning $(t = 0)$ will remain constant.



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