Time and frequency response of an oscillator

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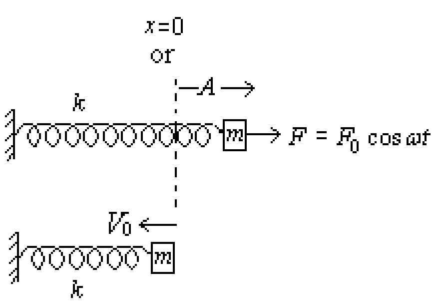
Abstract: In this paper we tried to explain the basics of response (displacement amplitude and velocity amplitude or potential and kinetic energy amplitudes) of the oscillator with respect to time and frequency of the driving agent; Near the resonance these responses are described in depth to derive the frequency range at which the oscillator responds strongly to certain driving frequencies. Then, the relation between the time and frequency is explained which forms the basis of understanding Heisenberg's uncertainty principle. At last, the time & frequency response of energy at resonant frequency is clearly explained with proper micro-interpretation in terms of energy. The key point is that both KE and PE (or velocity and displacement) response will not take place at same frequency which is critically analysed.

1. Response of a damped forced oscillator:

Let the block of mass m in a damped oscillator move under the action of a periodic force $F = F_0 \cos \omega t$, where time t is calculated from the undeformed position of the spring for the motion of the block along $+x$ -direction. Let $A =$ amplitude of oscillation, which is also a function of frequency ω given as

$$A(\omega) = \frac{F_0/\omega}{\sqrt{\left(m\omega - \frac{k}{\omega}\right)^2 + b^2}}$$

$$= \frac{F_0/m}{\sqrt{\left(\omega^2 - \omega_0^2\right)^2 + \left(\frac{b\omega}{m}\right)^2}}$$



Oscillating blcok by a driving force $F = F_0 \cos \omega t$

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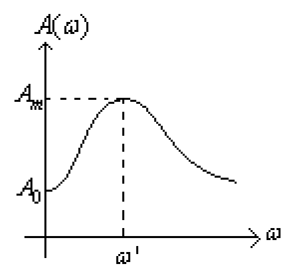
When $\omega \to 0$, $A = A_0 = \frac{F_0}{m\omega_0^2} = \frac{F_0}{k} \left(\because k = m\omega_0^2\right)$



Displacement amplitude varies with frequency in steady state

Putting, $F = F_0 \cos \omega t$, we have,

$m\frac{d^2x}{dt^2} + kx = F_0 \cos \omega t$ --- (i)

This differential equation is in-homogenous and linear[1]

Its general solution is given as $x = x_1 + x_2$ ---- (ii), where

$x_1 =$ general solution of the homogeneous equation

$\frac{d^2x}{dt^2} + \omega_0^2 x = 0$, $x_2 =$ Particular integral of the in-homeogeneous equation, $F = F_0 \cos \omega t$. x_1 and x_2 are given as

$x_1 = C_1 \cos(\omega_0 t + \phi)$ --- (iii)

$x_2 = C_2 \cos \omega t$ --- (iv)

Putting x_1 from eq.(iii) x_2 from (iv) in eq.(ii), we have

$x = C_1 \cos(\omega_0 t + \phi) + C_2 \cos \omega t$ --- (v)

Putting $(x = x_2)$ from eq.(iv) in eq.(i), we have

$$-mC_2\omega^2 \cos\omega t + kC_2 \cos\omega t = F_0 \cos\omega t$$

Or, $C_2 = \frac{F_0 / m}{\omega_0^2 - \omega^2}$, where $\omega_0 = \sqrt{\frac{k}{m}}$ --- (vi)

Putting C_2 from eq. (vi) in eq.(v), we have

As ω increases *A* will increase, at a particular $\omega = \omega'$ (say), *A* attains the maximum value and then it decreases to zero as ω tends to infinity. For maximum amplitude, $\frac{dA(\omega)}{d\omega} = 0$.

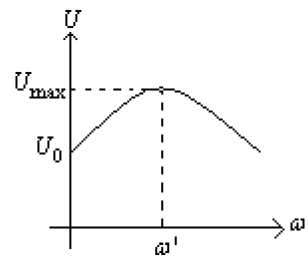
This gives, $\omega(=\omega') = \sqrt{\omega_0^2 - \frac{b^2}{2m^2}}$. For weak damping isnce $b \cong 0$; we have $\omega \cong \omega_0$[1]. This can also be compared with the frequency of damped oscillation of the block (if we remove the external force), which is given as $\omega = \sqrt{\omega_0^2 - \frac{b^2}{4m^2}}$.

From the last two expressions we can understand the effects of damping (friction) on the frequency of damped oscillation. The friction decreases the frequency of oscillation from natural frequency ω_0 or undamped frequency $\left(\omega_0 = \sqrt{\frac{k}{m}}\right)$ to $\omega' = \sqrt{\omega_0^2 - \frac{b^2}{4m^2}}$.

However, when we apply a periodic driving force $F = F_0 \cos\omega t$, the presence of damping reduces the transients (initial damping oscillations) to attain a steady state with oscillation frequency which is equal to the frequency of the driving force. This is the speciality of the forced oscillation. If we go on varying the frequency of the driving force so slowly that the oscillation becomes steady at each point of time and we can see the variation of amplitude with the varying frequency as given earlier. Putting $\omega = \sqrt{\omega_0^2 - \frac{b^2}{2m^2}}$ in the expression of $A(\omega)$, we have the maximum amplitude.

$$A(\omega)\big|_{max} = \frac{F_0}{2m\sqrt{\omega_0^2 - \frac{b^2}{2m^2}}}$$

When *b* increases, value of $\omega' \& \omega$ deceases. As a result, the peak of the curves will be shifted towards the origin. This phenomenon in which the steady state amplitude of a forced oscillator becomes maximum is called displacement (or amplitude) resonance. This occurrs at a frequency $\omega (< \omega_0)$ which is slightly less than the natural frequency of the oscillator for weak damping. At amplitude resonance, the elastic potential energy stored in the spring will be maximum.



U is maximum at $\omega = \omega'$

The potential energy verses the angular frequency can be given on $U = \frac{1}{2}KA^2$

$$= \frac{1}{2}\left(m\omega_0^2\right)\left[\frac{F_0 / m}{\sqrt{\left(\omega - \omega_0^2\right)^2 + \left(\frac{b\omega}{m}\right)^2}}\right]^2,$$

Or, $$U(\omega) = \frac{F_0^2 \omega_0^2}{2m\left[\left(\omega^2 - \omega_0^2\right)^2 + \frac{b^2\omega^2}{m^2}\right]}$$

The maximum potential energy is given by putting $\omega = \omega' = \sqrt{\omega_0^2 - \frac{b^2}{2m^2}}$ in this expression of $U(\omega)$.

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Then,

$$U(\omega)\big|_{\max} = \frac{F_0^2}{4\left(\omega_0^2 - \dfrac{b^2}{2m^2}\right)}$$

The potential energy at $\omega = 0$ is obtained by putting $\omega = 0$ in the expression of $U(\omega)$.

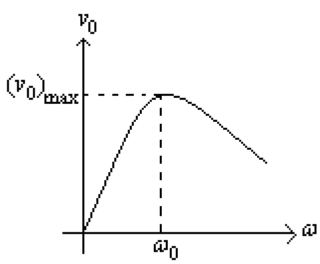
Then, $U_0 = \dfrac{F_0^2}{2m\omega_0^2}$

The velocity amplitude of the oscillating block will be $v_0 = A\omega$

$$= \frac{F_0/\omega}{\sqrt{\left(m\omega - \dfrac{k}{\omega}\right)^2 + b^2}}$$

Or $v_0 = \dfrac{F_0}{\sqrt{\left(m\omega - \dfrac{k}{\omega}\right)^2 + b^2}}$

or $v_0(\omega) = \dfrac{(F_0/m)\omega}{\sqrt{\left(\omega^2 - \omega_0^2\right)^2 + \left(\dfrac{b\omega}{m}\right)^2}}$

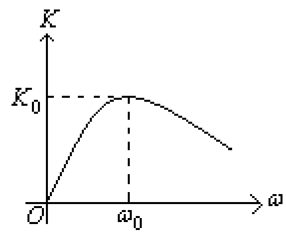


v_0 is maximum at $\omega = \omega_0$

Then, the velocity amplitude increases from zero at $\omega = 0$ to a maximum value where $\dfrac{dv_0(\omega)}{d\omega} = 0$, at $\omega = \omega_0$ and then decrease to zero when $\omega \to \infty$. The frequency $\omega = \omega_0$, at which velocity becomes maximum is called velocity resonance. The maximum velocity is obtained by putting $\omega = \omega_0$ in the expression of $v_0(\omega)$

$$v_0\big|_{\max} = \frac{F_0}{b\omega_0}$$

The maximum kinetic energy of the oscillating block m in steady state can be given as the function of frequency ω of oscillation as,



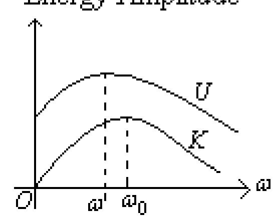
K.E is maximum at $\omega = \omega_0$

$$K(\omega) = \frac{1}{2} m v_0^2$$

$$= \frac{1}{2} m \left[\frac{F_0^2}{\left(m\omega - \dfrac{K}{\omega}\right)^2 + b^2} \right]$$

$$K(\omega) = \frac{F_0^2 \omega^2}{2m\left[\left(\omega^2 - \omega_0^2\right)^2 + \dfrac{b^2\omega^2}{m^2}\right]}$$

Energy Amplitude



Resonance of U & K takes place at $\omega = \omega'$ and $\omega = \omega_0$ respectively

The maximum value of kinetic energy amplitude is $K(\omega)_{\max} = \dfrac{mF_0^2}{2b^2}$. The kinetic energy amplitude becomes maximum at $\omega = \omega_0$, exactly at the natural frequency unlike potential energy resonance.

Hence, we should note that both resonances such as displacement (or potential energy) and velocity (or kinetic energy) do not take place at same frequencies, but nearly at same frequencies if we take a small damping constant "b".

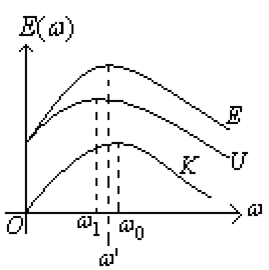
In other words, "kinetic energy resonates at $\omega = \omega_0$ and potential energy resonates at $\omega = \sqrt{\omega_0^2 - \frac{b^2}{2m^2}}\,(<\omega_0)$. The fact that K.E & P.E resonate at different frequencies is closely connected with the circumstance that a damped oscillator is not a conservative system and the energy is being constantly exchanged with a driving mechanism and is being transformed to the damping medium[2]". Since both kinetic and potential energy oscillate with respect to time with a frequency $\omega'(=2\omega)$, their average values are equal to half of their maximum values. Hence, the total average energy can be given as

$$E_{av} = U_{av} + K_{av}$$

$$= \frac{1}{2}U_{\max}(\omega) + \frac{1}{2}K_{\max}(\omega)$$

$$= \frac{1}{2}\left[U_{\max}(\omega) + K_{\max}(\omega)\right]$$

$$= \frac{1}{2}\left\{\frac{F_0^2\omega_0^2}{2m\left[\left(\omega^2-\omega_0^2\right)^2 + \frac{b^2\omega^2}{m^2}\right]} + \frac{F_0^2\omega^2}{2m\left[\left(\omega^2-\omega_0^2\right)^2 + \frac{b^2\omega^2}{m^2}\right]}\right\}$$



Total energy resonance takes place at $\omega = \omega'$ lying between ω_1 and ω_0

$$E_{av} = \frac{F_0^2\left(\omega_0^2 + \omega^2\right)}{4m\left[\left(\omega^2-\omega_0^2\right)^2 + \frac{b^2\omega^2}{m^2}\right]}$$

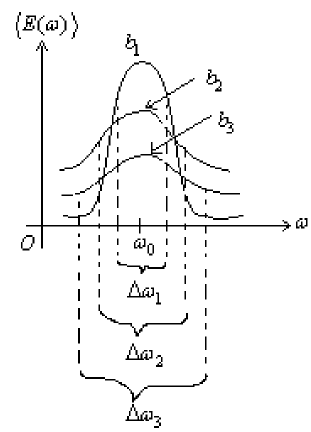
The total energy amplitude or its average value is maximum when $\frac{dE(\omega)}{d\omega} = 0$ and $\frac{d^2E(\omega)}{d\omega^2} = 0$

Then, $\omega_2 = \omega_0\sqrt{\sqrt{4 - \frac{b^2}{m^2\omega_0^2}} - 1}$

For weak damping $\omega \cong \omega_0$. For low damping and near resonance, we can write $\omega \cong \omega_0$ and hence the expression of $\langle E(\omega)\rangle$ can be simplified as

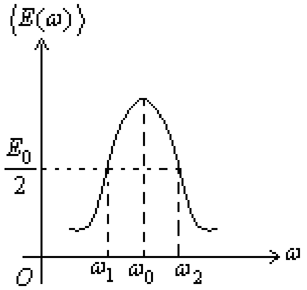
$$\langle E(\omega)\rangle = \frac{F_0^2}{m\left[\left(\omega-\omega_0\right)^2 + \frac{b^2}{4m^2}\right]}$$

The plot of the function $\left[(\omega-\omega_0)^2 + \left(\frac{b}{2m}\right)^2\right]^{-1}$, which contains the entire frequency dependence of $\langle E(\omega)\rangle$ is called a resonance curve or Lorentzian [3].



$\Delta\omega_1 < \Delta\omega_2 < \Delta\omega_3$
because $b_1 < b_2 < b_3$

For different values of b different peak values of $\langle E(\omega)\rangle$ are given. The energy of the oscillator will be half of the resonance energy peak if $\langle \omega-\omega_0 \rangle^2 = \left(\frac{b}{2m}\right)^2$ or $\omega-\omega_0 = \pm\frac{b}{2m}$



$E \geq \frac{E_0}{2}$ if ω lies between ω_1 and ω_2

Then, the frequency difference $\Delta\omega = \omega_2 - \omega_1$ at which stored energy will be half of the resonance energy can be called "band width" and is given as $\Delta\omega = \omega_2 - \omega_1 = 2(\omega-\omega_0)$ or $\Delta\omega = \frac{b}{m}$

As damping constant b decreases, $\Delta\omega$ will be lesser and hence, if $\frac{b}{m}$ decreases, the curve becomes higher and narrower, the range of frequency over which the system responds become smaller and the oscillator becomes increasingly selective in frequency[4]."

The above explanation keeps us informed that one of the important characteristic of an oscillator is to absorb energy from the driving agent to the maximum value within certain frequency range. This characteristic is called velocity of the oscillator. The frequency selective property of an oscillator is also be given by its quality factor or the Q-factor of the forced oscillator is defined as the ratio of energy stored in it in steady state and energy dissipated in it due to the friction in a time which corresponds to one radian, that is, $\Delta t = \frac{T}{2\pi}$.

Mathematically, $Q = \frac{\langle E_{\text{stored}} \rangle}{E_{spent} \text{ per radian}}$

$$= \frac{\frac{1}{4} mA^2(\omega^2+\omega_0^2)}{\langle bv^2 \rangle \times T/2\pi}$$

Near resonance,

$\omega \cong \omega_0$ and $v^2 = \langle v_0^2 \cos^2(\omega t - \phi) \rangle = \frac{v_0^2}{2}$

Then, $Q \cong \frac{mA^2\omega_0^3}{bv_0^2}$ or, $Q = \frac{m\omega_0}{b} (\because \omega_0 A = v_0)$

Or, $Q = \frac{\omega_0}{\omega_2 - \omega_1} \; (\because b = m\,\Delta\omega)$

If the Q-factor is more, $\Delta\omega$ will be less. Hence, the oscillator will be selective in a narrow range of frequencies near resonance. Thus, if Q is more the resonance peak will be sharper.

<u>Relation between the response in time & frequency</u>:

When damping is too small, an oscillator takes a long time to attain a steady state because the initial transient beats will die out after a long time. Once the oscillator attains a steady state; its amplitude will be large near resonance and hence it will store large energy. If we withdraw its driving force, it will again take a long time to dissipate all its stored energy after a large number of oscillations. In other words, Q-factor will be more. Further more, the oscillator will be more selective at low damping. This means that the oscillator will resonate with a quite narrow range of frequencies near the resonanance(or natural) frequency $\omega = \omega_0 = \sqrt{\frac{k}{m}}$.Hence, there is an intimate relation between frequency and time dependence (response) of energy. The decay time τ will be given as the time required to decrease (or increase) the energy by E_0/e, where E_0 = maximum energy stored in the oscillator. Since $E = E_0 e^{-\frac{bt}{m}}$, putting $E = \frac{E_0}{e}$ and $t = \tau$, we have, $\tau = \frac{m}{b}$.

We have already proved that the energy will be more if $\Delta\omega$ will be less which is given as $\Delta\omega = \frac{b}{m}$.

This states that lowering the damping, the frequency selectively increases. By using the last two equations, we have $\Delta\omega\tau = 1$. This means that the product of time of decay, that is, the time during which the oscillator gains (or loses) $\frac{1}{e}$ of the total energy and the frequency range in which the oscillator will possess more than half of the maximum (time averaged) energy, will be one. They $(\Delta\omega \,\&\, \tau)$ are inversely proportional to each other. Hence we cannot choose both $\Delta\omega \text{ and } \tau$ independently and arbitrarily. If we choose one $(\Delta\omega \text{ or } \tau)$, the other will be automatically fixed. In fact the above formula brings (relates) the two responses time and frequency of "energy" of the oscillator together[5].

In certain atomic system damping constant is extremely small and hence Q value $\left(=\frac{m}{b}\right)$ will be extremely large, say $Q = 10^8$. Then the atomic system can absorb significant energy in resonance only when the driving frequency is extremely close to its natural frequency. Hence, the atomic (or microscopic) systems remain affected by the external agents having many arbritrary frequencies. They can only be tuned by an extremely narrow band of frequency near resonance. Then, the frequencies of such oscillating system or "atomic clocks" are so accurate that they have superseded astronomical time standards. The above relation between frequency and time is extremely important in designing mechanical & electrical systems. "Any system which is highly frequency selective will oscillate for a long time if it is accidentally perturbed. Further more, such an element will take long time to reach the steady state when a driving force is applied because the effects of the initial conditions (oscillations) die out only slowly[6]." More generally the equation $\Delta\omega.\tau \cong 1$ can be written as $\Delta\omega.\,\Delta t = 1$

Multiplying both sides by $\frac{h}{2\pi}$, we have $\frac{h\Delta\omega}{2\pi}\Delta t = \frac{h}{2\pi}$

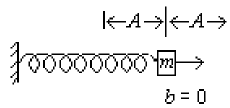
Or, $\Delta E.\Delta t \cong \tau$

This is indeed one of the form of Heisenberg's uncertainity principle which plays a fundamental role in quantum mechanics.

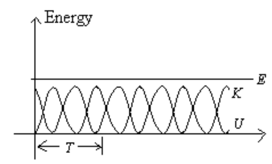
For instance of the time of excitation is $\Delta t = 10^{-8} s$, the radiated (or absorbed) energy band spreads in the range $\Delta\omega$ given as

$$\Delta\omega = \frac{h}{\Delta E} = \frac{6.63\times10^{-34}}{2\pi\times10^{-8}} \cong 1\times10^{-26} rad/s$$

Let us now examine the variation of total energy of a forced oscillator with respect to time near resonance. For the sake of simplicity let us assume that the block oscillates with a natural frequency $\omega_0 = \sqrt{\frac{k}{m}}$ and amplitude A with out any damping and external periodic force.



Undamped free oscillator oscillates with same amplitude A



In undamped free oscillation total energy of the oscillator remains constant

The total mechanical energy E of the oscillator is given by assuming $x = A\sin\omega t$. Then,

$$E = U + K$$

$$= \frac{1}{2}kx^2 + \frac{1}{2}mv^2$$

$$= \frac{1}{2}k(A\sin\omega_0 t)^2 + \frac{1}{2}m(A\cos\omega_0 t)^2$$

$$= \frac{1}{2}kA^2 \left(\because v = A\omega_0 \;\&\; m\omega_0^2 = K\right)$$

This means that, in a force undamped oscillator the total mechanical energy remains conserved be cause no external source is present to deliver the energy to the vibrating system. However, when we drive the block (or spring) acted upon by a periodic external force of frequency ω, at steady state, certain total mechanical energy will be stored in the system as the combination of kinetic and potential energy, given as

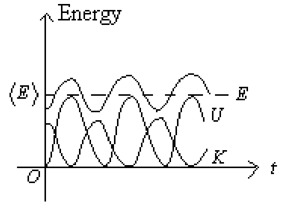
$$E = U + K$$

Or $E = \frac{1}{2}kx^2 + \frac{1}{2}mv^2$

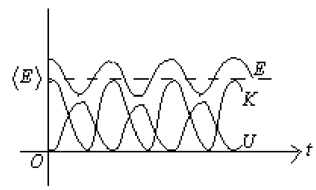
$$= \frac{1}{2}(m\omega_0^2)x^2 + \frac{1}{2}mv^2$$

Or $E = \frac{1}{2}m\left(\omega_0^2 x^2 + v^2\right)$

If $x = A\sin(\omega t - \phi)$ in steady state, we have, $v = \dfrac{dx}{dt} = \Delta\omega\cos(\omega t - \theta)$



since $\omega < \omega_0$
$K_{max} < U_{max}$



since $\omega > \omega_0$
$K_{max} > U_{max}$

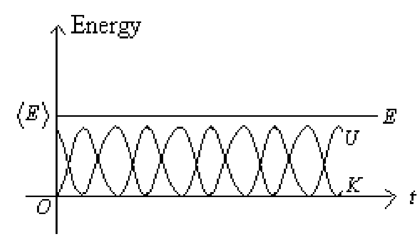
Then,

$$E = \frac{m}{2}\left[\omega_0^2 A^2 \sin^2(\omega t - \phi) + \omega^2 A^2 \cos^2(\omega t - \phi)\right]$$

Or, $E = \dfrac{mA^2}{2}\left[\omega_0^2 \sin^2(\omega t - \phi) + \omega^2 \cos^2(\omega t - \phi)\right]$

Or, $E = U_{max}\sin^2(\omega t - \phi) + K_{max}\cos^2(\omega t - \phi)$,

where $U_{max} = \dfrac{mA^2\omega_0^2}{2}$ and $K_{max} = \dfrac{mA^2\omega^2}{2}$



At $\omega = \omega_0$, $K + U$ = constant

For any frequency ω of the driving force, the amplitude of oscillation the block in steady state is a constant w.r.t time, given as

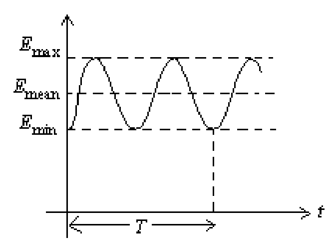
$$A = \frac{\frac{F_0}{m}}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + \left(\frac{b\omega}{m}\right)^2}}$$

We can see that, the ratio of maximum kinetic and potential energy of the oscillator is $\dfrac{K_{max}}{U_{max}} = \omega^2 : \omega_0^2$, where $\omega_0^2 = \dfrac{k}{m}$. If the driving frequency is not equal to the natural frequency $(\omega \neq \omega_0)$, we have unequal U_{max} and K_{max}. Hence, the total mechanical energy of the oscillator will oscillate between its maximum and minimum value about its mean value over time, that is, $\langle E\rangle = \dfrac{1}{4}m(\omega^2 + \omega_0^2)A^2$ because mean value of $\cos^2(\omega t - \phi)$ and $\sin^2(\omega t - \phi)$ over a time which is too longer than the period of oscillation of the driving force, is equal to $\dfrac{1}{2}$. The total mechanical energy of the oscillator at $\omega \neq \omega_0$ is given as

$$E = \frac{mA^2}{4}\left[\left(\omega_0^2 + \omega^2\right) + \left(\omega_0^2 - \omega^2\right)\cos 2(\omega t - \phi)\right]$$

This tells us that the total mechanical energy of the oscillator fluctuates about its mean value $\dfrac{mA^2}{4}\left(\omega_0^2 + \omega^2\right)$ with a frequency 2ω, double the frequency of oscillation of the force (or the block). The maximum and minimum energy of the oscillator can be obtained by putting $\cos 2(\omega t - \phi) = \pm 1$.

Then, $E_{max} = \dfrac{mA\omega^2}{2}$ and $E_{min} = \dfrac{mA\omega_0^2}{2}$



At $\omega \neq \omega_0$, the total energy vareis between E_{min} and E_{max}

For, $\omega \neq \omega_0$, the total energy fluctuates about its mean value. However, at $\omega = \omega_0$, E will not fluctuate with time. It remains constant over time, that is, $\dfrac{mA^2\omega_0^2}{2}$.

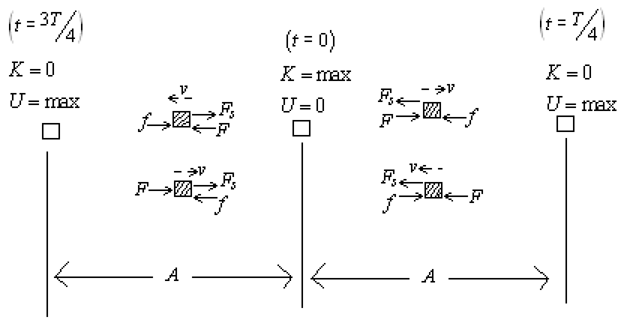
However, at $\omega \neq \omega_0$, E is a periodic function of time which changes at a rate of

$$\frac{dE}{dt} = \frac{mA^2\left(\omega^2 - \omega_0^2\right)\omega\sin 2(\omega t - \phi)}{2}$$

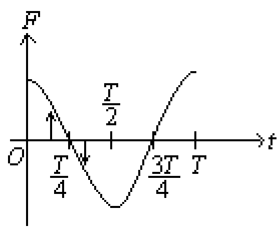
This can be positive, negative and zero at different instants.

Let us take four different snapshots at both sides of the mean position of the oscillating block. Since

$F = F_0 \cos \omega t$, during internal time from 0 to $T/4$ its magnitude decreases from F_0 to zero.



At all positions of the block, the driving force opposes friction



Variation of driving force with time

Directed to right, during the time internal from $T/4$ to $T/2$ its magnitude increases to maximum value F_0 and directed towards left and continues to point in same direction decreasing its magnitude from F_0 to zero at the extreme left during the time interval from $T/2$ to $3T/4$. Finally, it points towards right increasing its magnitude from 0 to F_0 at the mean position completing a cycle. We can see that the driving force always favors the velocity. Hence it always does positive work. Since friction always opposes the velocity, the driving force always opposes the friction. As friction opposes velocity, work done by friction is always negative. However, sometimes the driving force opposes and sometimes it favors the spring force, so that the net force acting on the block is the vector sum of $\vec{F}, \vec{f}, \vec{F}_s$ given as

$$\vec{F}_{net} = \left(\vec{F} + \vec{f} + \vec{F}_s\right) = \overrightarrow{ma}$$

Or, $\vec{F}\,\vec{v} + \vec{f}\,\vec{v} + \vec{F}_s\,\vec{v} = \overrightarrow{ma}.\vec{v}$

Or, $P_F + P_f - \frac{dU}{df} = +\frac{dK}{dt}$

Or, $P_F + P_f = \frac{d(U+K)}{dt}$

Or, $P_F + P_f = \frac{dE}{dt}$

The above expression states that the sum of power delivered by external force F and friction *f*, that is, the power delivered by the net force $\left(\vec{F} + \vec{f}\right)$ is numerically equal to the rate of change in mechanical energy of the oscillator.

Writing $P_F + P_f = (F - f)v$, and substituting the values of $\frac{dE}{dt}$ in the last expression, we have

$$(F - f)v = \frac{mA^2}{2}\left(\omega^2 - \omega_0^2\right)\omega \sin 2(\omega t - \phi)$$

This expression tells us that, when $\omega \neq \omega_0$, we have $\phi \neq 0$.

Some times $\frac{dE}{dt}$ is +ve, when $F > f$; $\frac{dE}{dt}$ is $-ve$, when $F < f$ and $\frac{dE}{dt} = 0$, when $\omega = 0$ or $F = f$ $\omega = \omega_0$ in other words, when the driving force does more work giving more energy to the system than the energy lost due to friction, the energy of the oscillator increases and vice versa.

If we put, $\vec{F} = -\vec{f}$ (the applied force is always equal and opposite to friction) in the vector equation $\vec{F} + \vec{f} + \vec{F}_s = \overrightarrow{ma}$, $\vec{F}_{sp} = m\vec{a}$ which means that the spring force is the net force providing necessary acceleration to the block. Putting $\left|F_{sp}\right| = kx$ and $\left|a\right| = \omega^2 x$, we have $\omega = \sqrt{\frac{K}{m}} = \omega_0$. Alternatively, putting in $(F - f)v = 0$ the equation $(F - f)v = \frac{mA^2}{2}\left(\omega^2 - \omega_0^2\right)\omega \sin 2(\omega t - \phi)$ we have $\omega^2 - \omega_0^2 = 0$ or $\omega - \omega_0 = 0$

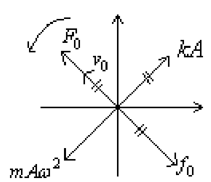
This means that, the "oscillator energy" will remain constant (or conserved) relative to time for a frequency ω of the driving agent which is equal to the natural (undamped forced oscillation) frequency of the given spring mass system.

$E_{osc} = \frac{mA^2\omega_0^2}{2}$,where

$$A = \frac{\frac{F_0}{m}}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + \left(\frac{b\omega}{m}\right)^2}} = \frac{F_0}{b\omega} (\because \omega = \omega_0)$$

The amplitude can also be calculated by writing $F_0 = f = bv_0 = bA\omega$. Then, $E_{osc} = \frac{mF_0^2}{2b}$ at $\omega = \omega_0$

Since the oscillator energy remains constant, the net work done by the non consecutive forces F and friction is zero. The external (driving) force just counteracts the friction, doing equal positive work which is just nullified by equal negative work done by friction at each instant. For this to happen, always the external force $\vec{F}$ and velocity $\vec{v}$ must change with zero phase difference, attaining their maximum or minimum value at same position and changing their directions at the extreme positions of the block. Hence, $k\vec{A} + mv_0 = 0 \left(\because \vec{F} + \vec{F}_s = 0\right)$



The driving force and friction are equalin magnitude and opposite in direction and phase

Since $\vec{F}_0 = -b\vec{v}_0$ and $\vec{F}_0 + \vec{F}_{s\,0} = 0$, we have F_0 and v_0 are parallel in phaser diagram, Further more $\phi = 0$, between force $\vec{F}$ and velocity $\vec{v}$. This means that driving force and velocity are parallel and in phase at $\omega = \omega_0$ (resonance). The driving force and friction are equal, opposite in direction and phase.

The power delivered by friction is

$P_f = fv = -bv.v = -bv^2$ Or,

$P_f = -bv_0^2 \cos^2(\omega t - \phi)$

The average value of P_f is given by putting

$\left\langle \cos^2(\omega t - \phi) \right\rangle = \frac{1}{2}$

Or, $\left\langle P_f \right\rangle = \frac{bv_0^2}{2}$

The energy dissipated by friction during the decay time τ is

$$\Delta E = \left\langle P_f \right\rangle \tau$$

$$= \frac{bv_0^2}{2}\tau = \frac{bv_0^2}{2}\frac{m}{b}\left(\because \tau = \frac{m}{b}\right)$$

$$= \frac{1}{2}mv_0^2 = \frac{1}{2}m\left(\omega_0 A\right)^2 = \frac{1}{2}mA^2\omega_0^2 = E_{stored}$$

This means that, at resonance $(\omega = \omega_0)$ the energy stored in the oscillator is numerically equal to the energy that is lost due to friction during one decay (mean) time of the oscillator. This is not true for $\omega \neq \omega_0$. In all the above discussions we assume weak damping (small values of b)

The power input on the oscillator is $P_{input} = Fv$

$$= F_0 \cos \omega t . v_0 \cos(\omega t - \phi)$$

$$= F_0 v_0 \left(\cos^2 \omega t . \cos\phi + \cos \omega t \sin \omega t \sin \phi\right)$$

Since the average value of $\cos^2 \omega t$ is $\frac{1}{2}$ and

$\sin \omega t \cos \omega t = 0$, we have,

$\left\langle P_{input} \right\rangle = \frac{F_0 v_0}{2}$

Since $|F_0| = |f_0| = bv_0$, we have

$\left\langle P_{input} \right\rangle = \frac{bv_0^2}{2} = \left\langle P_f \right\rangle$

This means that, in each unit time (or cycle) the energy supplied by the driving force is dissipated in friction, maintaining the average mechanical energy of the system constant, at $\omega = \omega_0$.

The average power input or average frictional loss is maximum at $\omega = \omega_0$ because maximum velocity amplitude (or velocity resonance) takes place at $\omega = \omega_0$.

Recapilutating, maximum power transfer but (not maximum energy stored) from the source to the oscillator takes place at $\omega = \omega_0$. When we with draw (turn off) the applied force, the maximum stored energy will dissipate during a time τ. Thus the average rate with which this energy dissipation takes place is numerically equal to the average work done by friction per second near resonance. However, this logic does not hold good for $\omega \neq \omega_0$ which is supported by the statement " since friction dissipates most of the energy in a time τ , the steady state stored energy is equal to that which has been supplied "recently" by the driving force, is within a time τ . Thus we expect that at equilibrium the stored energy will be approximately equal to the input power times τ , which is equal to the frictional power times τ . We have seen that is indeed the case for $\omega = \omega_0$ For ω not equal to ω_0 the relationship between input power and stored energy is less easily guessed[7]".

References:

1.University physics – ii, Anwar Kamal – damped vibrations – p – 376
2. University physics – ii, Anwar Kamal – damped vibrations – p – 379
3. An introduction to Mechanics, Kleppner et.al – p – 428
4. An introduction to Mechanics, Kleppner et.al – p - 432
5. An introduction to Mechanics, Kleppner et.al – p - 428
6. An introduction to Mechanics, Kleppner et.al – p - 432
7. Berkeley Physics Course – vol-3, waves; Frank S Crawford Jr – p - 107